Indian Statistical Institute, Bangalore Centre
J.R.F. (I Year) : 2016-2017
Semester I : Final Examination
Analysis - I

07.11.2016 Time: 3 hours. Maximum Marks : 60

Note: Notation and terminology are understood to be as used in class. State clearly the results you are using in your answers.

1. ($3 \times 5 = 15$ marks) Prove or disprove the following:

   (i) Let $(\Omega, \mathcal{B}, \mu)$ be a $\sigma-$finite measure space. Let $A_n \in \mathcal{B}$, $n = 1, 2, \cdots$. Then $\mu(\limsup_{n\to\infty} A_n) = \limsup_{n\to\infty} \mu(A_n)$.

   (ii) Let f_n, $n = 1, 2, \cdots$, f be real valued Borel measurable functions and
   on a $\sigma-$finite measure space $(\Omega, \mathcal{B}, \mu)$. If $f_n \to f$, $\mu-$a.e., then $f_n \to f$ in $\mu-$measure.

   (iii) Let μ be a finite measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ such that $\mu\pi_1^{-1}$, $\mu\pi_2^{-1}$ are both absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$; (here π_1, π_2 denote the coordinate projections on $\mathbb{R}^2$). Then μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^2$.

2. (10 marks) For $\alpha > 0$, define

$$\Gamma(\alpha) = \int_{(0,\infty)} x^{\alpha-1}\, e^{-x} dx,$$

   where dx denotes integration w.r.t. the Lebesgue measure on $\mathbb{R}$. Show that $\alpha \mapsto \Gamma(\alpha)$ is continuous on $(0, \infty)$.

3. (15 marks) Let μ, ν be finite measures on a measurable space $(\Omega, \mathcal{B})$. Put $\lambda = \mu - \nu$; let $\mid \lambda \mid$ denote the total variation measure of λ. For any bounded Borel measurable function f on $(\Omega, \mathcal{B})$, show that

$$\begin{aligned} \mid \int_\Omega f(\omega) d\lambda(\omega) \mid &\leq \int_\Omega \mid f(\omega) \mid d \mid \lambda \mid (\omega) \\ &\leq \int_\Omega \mid f(\omega) \mid d\mu(\omega) + \int_\Omega \mid f(\omega) \mid d\nu(\omega). \end{aligned}$$

4. (10 marks) Let $p, q \geq 1$ such that $(1/p) + (1/q) = 1$. Let μ denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Let $f \in L^p(\mu)$, $g \in L^q(\mu)$. Show that $\| f * g \|_\infty \leq \| f \|_p \| g \|_q$.

5. (10 marks) Let μ, ν be $\sigma-$finite measures on a measurable space $(\Omega, \mathcal{B})$ such that $\mu \ll \nu,\ \nu \ll \mu$. Show that $(L^1(\mu))^* = (L^1(\nu))^*$. Is $L^1(\mu) = L^1(\nu)$?